Red Level

1. (6 credits) How many different values of x satisfy the equation $\cos^2 x + 2\sin^2 x = 1$, provided that $-10 < x < 10$?

Solution. Since $\cos^2 x + \sin^2 x = 1$, the equation is equivalent to $\sin^2 x = 0$, and numbers divisible by π are its solutions. Since $3\pi < 10 < 4\pi$ and $-4\pi < -10 < -3\pi$, the numbers -3π, -2π, ..., 3π satisfy the double inequality, and this gives 7 values in total.

2. (6 credits) In McDonald's, the middle set of 3 hamburgers, 5 milk shakes and 1 pack of French fries costs 235 rubles, and the large set of 5 hamburgers, 9 milk shakes and 1 pack of French fries costs 395 rubles. How much does the small set of 2 hamburgers, 2 milk shakes and 2 packs of French fries cost there if all prices are determined correctly and without discounts?

(Answer: 150 rubles.)

Solution. Let G be the price of a hamburger, M be the price of a cocktail, and K be the price of a pack of French fries. It follows from the equations 3G+5M+K=235 and 5G+9M+K=395 that G+M+K=2(3G+5M+K)-(5G+9M+K)=470-395=75, therefore 2(G+M+K)=150.

3. (8 credits) Alik, Borya and Vasya solved 100 problems together, and each of them solved exactly 60 problems. Let the problem solved by all of them be called 'easy' one, and let the problem solved by only one boy be called 'difficult' one (each problem is solved by at least one boy). What number is greater: the number of easy problems or the number of difficult ones, and what is the difference between the greater number and the lesser number?

(Answer: The number of difficult problems exceeds the number of easy problems by 20.)

Solution. Let X be the number of easy problems, Z be the number of difficult ones, and Y be the number of problems solved by two participants. X+Y+Z=100, and 3X+2Y+Z=3*60 (this is the equation for the sum of 60 tasks solved by each of the three participants). Therefore, Z-X=2(X+Y+Z)-3X-2Y-Z=200-180=20.

4. (8 credits) The quadrilateral MOST can be inscribed in a circle. It is known that $\angle OMT=20°$, and $\angle MTS=100°$. Find the angle (in degrees) between the extensions of the sides MT and OS.

(Answer: 80.)

Solution. If a quadrilateral can be inscribed in a circle, the sum of its opposite angles is 180°, hence we can calculate the remaining two angles of the quadrilateral: $\angle MOS=80°$, $\angle OST=160°$. The extensions of the sides MT and OS form the triangle STP. Its angles S and T are considered external for the quadrilateral and they are equal to 180° less the internal angle of the quadrilateral, i.e. $\angle STP=180°-\angle MTS=80°$, $\angle TSP=180°-\angle OST=20°$. Therefore, the third angle of the triangle is $SPT=180°-\angle PST-\angle PTS=180°-20°-80°=80°$.

5. (10 credits) Vanya has three rectangular parallelepipeds, each having a volume of 128. The areas of the two faces of the first parallelepiped are 4 and 32, the areas of the two faces of the second parallelepiped are 16 and 64, and the areas of the two faces of the third parallelepiped are 8 and 32. What can be the maximum height of a tower constructed of these parallelepipeds?

(Answer: 56.)

Solution. Direct calculation gives us a possibility to find all dimensions: 1x4x32, 2x8x8 and 2x4x16. The maximum height of a tower is equal to the sum of the three largest dimensions, i.e. 32+16+8=56.

6. (10 credits) Numbers A, BC, DD and AAE (each digit is replaced by a letter, each letter corresponds to one digit, the same letters denote the same numbers) are terms of an increasing arithmetic progression. Find the digit denoted by C.

(Answer: 9.)

Solution. A=1, therefore AAE=112, 115 or 118 (the number AAE-1 must be divisible by three). The difference of the progression is equal to 37, 38 and 39 accordingly. The second value is the only one that gives a number consisting of the same digits (1+2*38=77) for the third term of the progression. In this case BC=1+38=39, therefore C=9.

7. (10 credits). Every station of the children's railway gives tickets to every other station. All these tickets are different, the start and end stations are indicated on each ticket. Some (more than one) stations were added on this railway recently, and 46 new types of tickets had to be printed additionally. How many stations are now operating on the children's railway?

(Answer: 13.)

Solution. If the initial number of the stations was x and then it became y, the current number of tickets required is y(y-1), and the previous one was x(x-1). According to the statement of the problem, y(y-1)-x(x-1)=46, i.e. (y-x)(y+x-1)=46. This means that y-x=2 and x+y=24. Accordingly, y=13 and x=11.

8. (12 credits) 2, 3, and 5 are roots of the polynomial f(x)=x^4+ax^2+bx+c. Find f(1).

(Answer: -88.)

Solution. Since we have three roots, the fourth one exists as well. The sum of all four roots is equal to the coefficient of x^3 multiplied by -1, i.e. it is equal to 0. This means that the fourth root is -10. Therefore, f(x)= (x−2)(x-3)(x−5)(x+10), and f(1)=(-1)·(-2)·(-4)·11=-88.

9. (15 credits) Let x♢y be a positive number defined as a function of x and y in accordance with some rule for any two positive numbers x and y. It is known that the operation ♢ satisfies the properties (x·y)♢y=x(y♢y) and (x♢1)♢x=x♢1 for all x, y > 0, and 1♢1=1. What is the value of 20♢17?

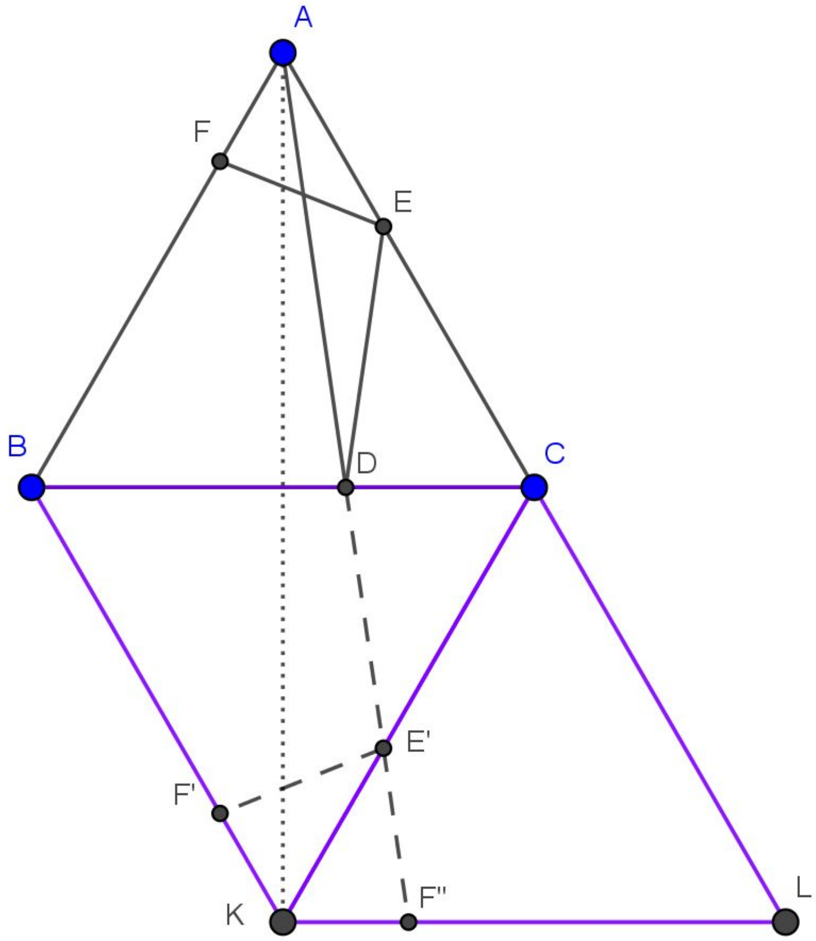
(Answer: 20.)

Solution. Let us prove that x♢y=x. First, this is true for y=1: x♢1=(x·1)♢1=x(1♢1)=x·1=x. Second, this is true for y=x (similar to the proof set out above): x♢x=(x·1)♢x=x♢1=x. Thus, according to the statement of the problem, (x·y)♢y=x (y♢y)=xy, and this means that the result of such operation is equal to the first 'multiplier'. After substituting xy for 20 and y for 17, we immediately find that 20♢17=20.

10. (15 credits) Vanya plays billiards on a table shaped as an equilateral triangle ABC with a side of 40 cm. He sends the ball from the vertex A in such way that the ball stops near the side AB 10 cm from the vertex A after two reflections from the walls BC and CA. What is the length of the path covered by the ball?

(Answer: 70 cm.)

Solution.



The ball's trajectory is the polygonal chain ADEF. We reflect the triangle ABC with the corresponding parts of the trajectory first across the side BC (in this case we get the triangle KBC), and then across the side KC (in this case we get the triangle KLC). Since the angle of incidence of the ball is equal to the angle of reflection from the wall, the polygonal chain "straightens" after two reflections into the segment AF". Now we have to find its length. Since KF"=KF'=AF=10 and AK= $40\sqrt{3}$, according to the theorem of Pythagoras $AF''^2=100+4800=4900$, therefore AF''=70.